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Flow field due to a row of vortex and source lines spanning a conical annular duct

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Abstract. Analytical expressions are derived for the incompressible flow due to a row of vortex and source lines spanning a duct whose walls are coaxial conical surfaces of revolution with a common vertex. The singularity lines have the shape of an arc of circle meeting the walls perpendicularly, and are defined by the intersection of a spherical surface with a series of equally spaced meridional planes. Although source lines of spanwisely varying strength are in general assumed, only vortex lines of constant circulation are considered. Simpler expressions are derived for the limiting two-dimensional cases when the flow is axisymmetric (actuator disc), and when the angular distance between the conical walls becomes vanishingly small. The expressions for the latter case are used in an example to obtain numerical results by a panel method for the velocity distribution of the flow about the inlet guide vane system of a water turbine of bulb type.

1. Introduction

One of the approaches to the theoretical analysis of the internal flow in turbomachinery consists in representing the presence of blades (rotating and stationary) and struts by singularities of the vortex, source, sink and dipole types. Two of the earliest and best known methods using singularities to describe the two-dimensional plane flow about a rectilinear cascade of aerofoil profiles are due to Schlichting [1] and Martensen [2] (for a comprehensive review see [3] and [4]). The fully analytical solution of the three-dimensional problem has been attempted only for relatively simple and idealized geometries. In the case of a rectilinear cascade bounded by plane parallel walls, solutions have been obtained for the flow about twisted blades spanning the walls normally [5] or with sweep and dihedral effects [6]. The incompressible flow about an annular cascade of radial source and vortex lines of constant strength bounded by coaxial cylindrical walls was studied by Meyer [7] (see also [8]) who integrated Laplace's equation for the potential by separation of variables. This solution was extended by McCune [9] to source lines of radially varying strength in linearized compressible flow. In the case of lifting lines of spanwisely varying circulation, the problem is made more difficult by the presence of trailing vortices convected by the flow, as is well known in propeller theory (see e.g. [10]), and analytical solutions can in practice be obtained only if simplifying assumptions are introduced concerning the shape of the convected vortex filaments. In the case of a rotor in an annular cylindrical duct, the usual assumption consists in taking the free vortex lines to be of truly helical shape, building together helical vortex surfaces (small perturbation theory). Solutions for this problem were obtained by the present author [11] for incompressible flow and by Okurounmu and McCune [12] for linearized compressible flow.

Among the great variety of bladed ducts occurring in turbomachinery, it is not uncommon to find cases when the walls can be adequately modelled by conical surfaces, with the blades' axes meeting the end walls at approximately right angles. As an example, we mention the inlet guide vane system of certain hydraulic turbines of bulb type (see e.g. [13]). In the present paper, theoretical results are derived for this kind of situation, with the lifting and thickness effects of the blades represented by vortex and source lines respectively. The flow is assumed incompressible, and irrotational outside the vortex singularities. In order to obtain exact analytical solutions, we further restrict the geometry of the inner and outer walls to conical surfaces of revolution with a common vertex, which, in a system of spherical co-ordinates (r, θ, ϕ), are given by $\theta = \text{constant}$. The vortex and source lines are circular arcs defined by $r = \text{constant}$, $\phi = \text{constant}$. The solution of Laplace's equation for the velocity potential with the required boundary conditions is achieved by the method of separation of variables. Unlike in the case of screw propellers and other types of open turbomachines, constant blade circulation along the span is widely adopted as a design condition in closed turbomachines, which means a change in (circumferentially averaged) tangential velocity that is inversely proportional to the distance from the axis (constant blade work for a rotor). On the other side, lifting-lines of spanwisely varying circulation imply the presence of trailing vortex sheets, which in conical flow, even assuming small perturbation, are of considerably more complex shape than the helical sheets in cylindrical flow for which the solutions mentioned above were obtained. For these reasons, only lifting-lines of constant circulation are considered here. Such restriction does not apply to the source lines, which are assumed here in general to be of varying strength along the span.

In Section 3, we look for the form taken by the general expressions, derived in Section 2 for the three-dimensional case, when two limiting situations are reached in which the flow becomes two-dimensional. The first one (Section 3.1) is the axisymmetric flow resulting from taking the circumferential average of the velocity field, or equivalently the flow due to an actuator disc made up of vortex or source lines. The second case (Section 3.2) concerns a conical duct for which the angular distance between the walls becomes vanishingly small and the flow reduces to an infinitely thin layer whose thickness is proportional to the distance from the vertex. The expressions derived for the latter case are used, in Section 3.3, to obtain alternative expressions for the potential of the fully three-dimensional case, in the form of a superposition of an essentially two-dimensional potential and a singularity-free additional field representing the three-dimensionality.

Section 4 deals with the application of the analytical expressions to turbomachinery flow problems. In particular it is shown how they can be employed to obtain numerical results for the velocity field of the flow through the conical-walled inlet guide vane system of a water turbine.

2. Three-dimensional analysis

We consider the incompressible inviscid flow through a cascade of N equally spaced vortex lines or source lines spanning two conical walls, and choose a system of spherical co-ordinates (r, θ, ϕ), with unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$. The walls are at $\theta = \theta_1$, $\theta = \theta_2$ $(0 < \theta_1 < \theta_2 < \pi)$. The singularity lines are defined by $r = 1$, $\theta_1 \leq \theta \leq \theta_2$, $\phi = 2\pi n/N$ $(n = 0, 1, 2, \ldots, N-1)$, their form being therefore an arc of circle (Fig. 1). The flow field is assumed irrotational everywhere except at the vortex lines and at the origin. We note that the radial flow, $V_r = \text{constant}/r^2$, $V_\theta = V_\phi = 0$, due to a source at the origin, and the swirling flow, $V_\phi = \text{constant}/(r \sin\theta)$, $V_r = V_\theta = 0$, due to a vortex along the axis of symmetry, both satisfy the

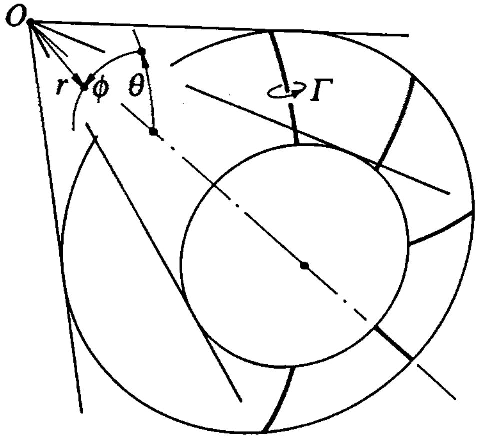

Fig. 1. Geometry of the duct and singularity lines, showing the system of spherical co-ordinates.

conditions of irrotationality and incompressibility, as well as the boundary condition at the conical walls, and so can be used to build up solutions by superposition.

2.1. Cascade of vortex lines

In the following, we will assume the vortex lines to be of constant strength Γ along the span. Since there is no shed vorticity, the solution for the velocity field will apply to a stationary as well as a rotating row of lifting lines. Using a result for a more general case [5], we obtain the following expression for the velocity field $\mathbf{V}$:

$$\mathbf{V} = \nabla\Phi - \mathbf{e}_\phi H(r-1)\frac{\Gamma}{r\sin\theta}\sum_{n=-\infty}^{\infty}\delta\left(\phi - \frac{2\pi n}{N}\right), \tag{1}$$

where Φ is a velocity potential, $H(\ )$ is Heaviside's unit step function, $\delta(\ )$ is Dirac's delta function, and $\mathbf{e}_\phi$ is the tangential unit vector. We note that, since Φ is a discontinuous function (see below), the differential operators ∇ and ∇^2 are to be taken as defined in the theory of generalized functions (see e.g. [14]). Introducing the equation of continuity, $\nabla\cdot\mathbf{V} = 0$, the following equation is found for Φ:

$$\nabla^2\Phi = H(r-1)\frac{\Gamma}{r^2\sin^2\theta}\sum_{n=-\infty}^{\infty}\delta'\left(\phi - \frac{2\pi n}{N}\right). \tag{2}$$

The flow region external to the N vortex lines is multiply-connected, with degree of connectivity $N+2$; if a continuous velocity-potential is defined in this region it is a many-valued function, with cyclic constant Γ for circuits looping a vortex line. Instead we define Φ in the region external to the N vortex lines and having as barriers the meridional half-planes $r > 1$, $\phi = 2\pi n/N$ $(n = 0, 1, 2, \ldots, N-1)$; this is now a doubly-connected region (it is not singly-connected due to the presence of the conical hub). We introduce the additional condition that, for any circuit looping the conical wall in the region $r < 1$, the cyclic constant of $\nabla\Phi$ has the value $N\Gamma/2$ (i.e. equals the circulation induced by a vortex along the duct's axis of strength $N\Gamma/2$). It is not difficult to conclude that Φ is discontinuous across the N meridional half-planes $r > 1$, $\phi = 2\pi n/N$ $(n = 0, 1, 2, \ldots, N-1)$, with jumps equal to Γ. Since it is more convenient to deal with a single-valued continuous function in the regions $r<1$ and $r>1$, we write

$$\Phi = \Phi_1 - \Gamma H(r-1)S(\phi) + \frac{N\Gamma\phi}{4\pi}, \tag{3}$$

where

$$\begin{aligned} S(\phi) &= \frac{1}{2} + \frac{N\phi}{2\pi} - H(\phi) - \sum_{n=1}^{\infty}\left[H\left(\phi - \frac{2\pi n}{N}\right) - H\left(-\phi - \frac{2\pi n}{N}\right)\right] \\ &= \frac{1}{\pi}\,\mathrm{Re}\left\{\sum_{n=1}^{\infty}\frac{\mathrm{i}}{n}\exp(\mathrm{i}nN\phi)\right\} \end{aligned} \tag{4}$$

is a saw-tooth function of period $2\pi/N$ varying linearly between $-1/2$ and $1/2$ in the intervals $2\pi n/N < \phi < 2\pi(n+1)/N$ ($n = 0, \pm 1, \pm 2, \ldots$). Defined in this way, Φ_1 is single-valued, continuous and continuously differentiable in the whole space, except at $r=1$, and is found to satisfy Laplace's equation $\nabla^2\Phi_1 = 0$ ($r \neq 1$).

The problem of obtaining suitable solutions for the potential Φ_1 can be dealt with by the method of separation of variables in our spherical system of co-ordinates [15]. Denoting the inner flow ($r<1$) and the outer flow ($r>1$) by the superscripts (i) and (o) respectively, we find the following double series of eigenfunctions

$$\begin{bmatrix}\Phi_1^{(i)} \\ \Phi_1^{(o)}\end{bmatrix} = \mathrm{Re}\left\{\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\begin{bmatrix}A_{np}^{(i)} \\ A_{np}^{(o)}\end{bmatrix} r^{\pm\alpha_{np}-1/2}\,\mathrm{e}^{\mathrm{i}nN\phi}T_{np}(y)\right\}, \tag{5}$$

where $y = \cos\theta$, and $A_{np}^{(i)}$, $A_{np}^{(o)}$ are constants to be determined from the matching conditions at $r=1$ to be discussed later. In (5), and throughout the present paper, the upper sign, in $\pm$ or $\mp$, is to be taken for the inner flow ($r<1$) and the lower sign for the outer flow ($r>1$). The spherical eigenfunctions $T_{np}(y)$ and the eigenvalues α_{np} (taken as nonnegative) are defined by Legendre's associated equation

$$(1-y^2)\frac{\mathrm{d}^2T_{np}}{\mathrm{d}y^2} - 2y\frac{\mathrm{d}T_{np}}{\mathrm{d}y} + \left[\alpha_{np}^2 - \frac{1}{4} - \frac{n^2N^2}{1-y^2}\right]T_{np} = 0, \tag{6}$$

together with the boundary conditions $T'_{np}(y_1) = T'_{np}(y_2) = 0$ (arising from the condition of zero normal velocity at the conical walls) and the normalizing condition

$$\int_{y_2}^{y_1} T_{np}^2(y)\,\mathrm{d}y = y_1 - y_2 \quad (y_1 = \cos\theta_1,\ y_2 = \cos\theta_2). \tag{7}$$

The expressions of the functions $T_{np}(y)$ are then found to be linear combinations of Legendre's associated functions of the first and second kind [16],

$$T_{np}(y) = C_{np}P^{nN}_{\alpha_{np}-1/2}(y) + D_{np}Q^{nN}_{\alpha_{np}-1/2}(y),$$

and the eigenvalues α_{np} are the roots of the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}y}P^{nN}_{\alpha-1/2}(y)\right]_{y_1}\left[\frac{\mathrm{d}}{\mathrm{d}y}Q^{nN}_{\alpha-1/2}(y)\right]_{y_2} - \left[\frac{\mathrm{d}}{\mathrm{d}y}P^{nN}_{\alpha-1/2}(y)\right]_{y_2}\left[\frac{\mathrm{d}}{\mathrm{d}y}Q^{nN}_{\alpha-1/2}(y)\right]_{y_1} = 0. \tag{8}$$

The values of the coefficients C_{np}, D_{np} can be determined from $T'_{np}(y_1) = 0$ or $T'_{np}(y_2) = 0$, and from (7). It can be shown that, for each n ($n = 0, 1, 2, \ldots$), the functions $T_{np}(y)$ ($p = 0, 1, 2, \ldots$) form a complete set of orthogonal functions with weighing function equal to a constant. In particular, $\alpha_{00} = \frac{1}{2}$ and $T_{00} = 1$. If we multiply (6) by T_{np}, integrate between y_2 and y_1, and take into account the boundary conditions, we obtain

$$\alpha_{np}^2 - \frac{1}{4} = (y_1 - y_2)^{-1} \int_{y_2}^{y_1} \left[(1 - y^2) \left(\frac{\mathrm{d}T_{np}}{\mathrm{d}y} \right)^2 + \frac{n^2 N^2 T_{np}^2}{1 - y^2} \right] \mathrm{d}y \,. \tag{9}$$

Therefore $\alpha_{np} \geqslant 1/2$, the equality sign applying only to α_{00}. It follows that the gradient of Φ_1, as given by (5), vanishes as $r \to 0$ or $r \to \infty$.

Expressions can be found for the constants $A_{np}^{(i)}$ and $A_{np}^{(o)}$ by introducing the condition that the potential Φ and its r-derivative should be continuous across the cascade surface $r = 1$. We then obtain $A_{0p}^{(i)} = A_{0p}^{(o)} = 0$ ($p = 0, 1, 2, \ldots$) and

$$\begin{bmatrix} A_{np}^{(i)} \\ A_{np}^{(o)} \end{bmatrix} = \mp \frac{\mathrm{i}\Gamma}{2\pi n} t_{np} \left(1 \pm \frac{1}{2\alpha_{np}} \right), \quad n = 1, 2, \ldots, \quad p = 0, 1, 2, \ldots, \tag{10}$$

where

$$t_{np} = (y_1 - y_2)^{-1} \int_{y_2}^{y_1} T_{np}(y) \, \mathrm{d}y \,, \qquad \sum_{p=0}^{\infty} t_{np} T_{np}(y) = 1 \,. \tag{11}$$

We note that the circumferentially averaged value of $\nabla\Phi_1$ is zero, and so Φ_1 does not contribute to the swirl circulation (about a closed curve enclosing the inner duct) in either the inner ($r < 1$) or the outer ($r > 1$) flow field. It follows that, for the velocity field as given by (1) and (3), the swirl circulation induced by the N lifting lines is evenly divided between the inner and outer flow regions, and is found to be equal to $\pm N\Gamma/2$ for $r \lessgtr 1$.

A difficulty arises if these results are to be used for numerical evaluation or in a lifting-surface theory, due to the singular behaviour, at the vortex lines, of the double series in (5), and its r- and ϕ-derivatives, implying poor convergence near those lines. Two kinds of singularities are expected to occur in the velocity field. The first one is essentially two-dimensional and is of the type (distance)$^{-1}$. The second one is associated with the lines' curvature; Küchemann and Weber [17] have shown that this singularity is logarithmic and affects the circumferential velocity component V_ϕ. For given θ, the singularities of both kinds remain essentially unchanged if the conical walls θ_1 and θ_2 are replaced by walls at θ and $\theta + \mathrm{d}\theta$, in which case the flow can be dealt with more simply as two-dimensional; in Subsection 3.2 a solution is derived for this case. Then in Subsection 3.3 we obtain an alternative expression for the fully three-dimensional flow as the superposition of the two-dimensional solution and a singularity-free double series accounting for the three-dimensionality.

2.2. *Cascade of source lines*

Let q be the volume flow rate emitted by each source line per unit length of line. We assume q in general to be a function of θ (or of $y = \cos\theta$). The total flow rate emitted by one source line is

$$Q = \int_{\theta_1}^{\theta_2} q(\theta)\,\mathrm{d}\theta\,. \tag{12}$$

With the velocity field given by $\mathbf{V} = \nabla\Phi$, the equation of continuity can be written as

$$\nabla^2\Phi = \frac{q}{\sin\theta}\,\delta(r-1)\sum_{n=-\infty}^{\infty}\delta\left(\phi - \frac{2\pi n}{N}\right). \tag{13}$$

We write the potential in the following form

$$\Phi = \frac{NQ}{4\pi(y_1 - y_2)}\left|\frac{1}{r} - 1\right| + \Phi_1\,. \tag{14}$$

The first term on the right-hand side of (14) represents, for $r<1$ or $r>1$, the flow due respectively to a point sink or a point source at the origin, absorbing or emitting through the duct a flow rate equal to $NQ/2$. Consequently the second term, Φ_1, is required to give no net contribution to the total flow rate through any duct section $r = \text{constant}$ for either $r<1$ or $r>1$.

The potential Φ_1 satisfies Laplace's equation (for $r \neq 1$), which can be solved by separation of variables as before, leading again to expression (5). By requiring the potential Φ to be continuous across $r = 1$, we obtain $A_{np}^{(i)} = A_{np}^{(o)}$ $(\equiv A_{np})$. The values of the coefficients A_{np} will be determined from the equation of continuity, as follows. Expressing $\nabla^2\Phi$ in spherical co-ordinates, multiplying both sides of (13) by r^2 and integrating between $r = 0$ and $r = \infty$, we find

$$\left[r^2\frac{\partial\Phi}{\partial r}\right]_{r=0}^{\infty} + \int_0^{\infty}\left[\frac{\partial}{\partial y}\left((1-y^2)\frac{\partial\Phi}{\partial y}\right) + (1-y^2)^{-1}\frac{\partial^2\Phi}{\partial\phi^2}\right]\mathrm{d}r$$
$$= (1-y^2)^{-1/2}q(y)\sum_{n=-\infty}^{\infty}\delta\left(\phi - \frac{2\pi n}{N}\right). \tag{15}$$

The summation in (15) represents a periodic delta function. An expression for it can be obtained by taking the ϕ-derivative of (4) and is

$$\sum_{n=-\infty}^{\infty}\delta\left(\phi - \frac{2\pi n}{N}\right) = \frac{N}{2\pi} + \frac{N}{\pi}\sum_{n=1}^{\infty}\cos(nN\phi)\,. \tag{16}$$

It is convenient to express $(1-y^2)^{-1/2}q(y) \equiv q(y)/\sin\theta$, on the right-hand side of (15), as a series of the orthogonal eigenfunctions $T_{np}(y)$,

$$\frac{q(y)}{\sin\theta} = \sum_{p=0}^{\infty} a_{np}T_{np}(y)\,, \quad n = 0, 1, 2, \ldots, \tag{17}$$

with

$$a_{np} = (y_1 - y_2)^{-1}\int_{y_2}^{y_1}\frac{q(y)}{\sin\theta}\,T_{np}(y)\,\mathrm{d}y\,. \tag{18}$$

It turns out that $a_{00} = Q/(y_1 - y_2)$. We note also that, if $q/\sin\theta = \text{constant}$, then $a_{0p} = 0$ $(p = 1, 2, \ldots)$; this result, that will be used later in the paper, follows from $T_{00} = 1$ and the fact that any constant is orthogonal to the remaining eigenfunctions T_{0p}, $p = 1, 2, \ldots$. Taking into account (14), (5) and (17), and performing the integration indicated in (15), we find $A_{00} = 0$,

$$A_{0p} = -\frac{N}{4\pi}\frac{a_{0p}}{\alpha_{0p}}, \quad p = 1, 2, \ldots, \tag{19}$$

$$A_{np} = -\frac{N}{2\pi}\frac{a_{np}}{\alpha_{np}}, \quad n = 1, 2, \ldots, \quad p = 0, 1, 2, \ldots. \tag{20}$$

Like what was said above in connection with the vortex lines, singularities of the same type are expected in the velocity field in the case of source lines; the main difference lies in that the logarithmic singularity affects the radial velocity component V_r rather than V_ϕ (see [17]). An alternative expression for the flow field is derived in Subsection 3.3.

3. Two-dimensional analyses

In the practical calculations of turbomachinery, one of the most frequently adopted simplifying assumptions consists of considering the three-dimensional flow field as the combination of two kinds of two-dimensional fields. One is the axisymmetric field (the so-called meridional flow) which results from taking the circumferential average of the flow velocity. The other one is the blade-to-blade flow on the stream surfaces of revolution of the meridional flow, or, more precisely, between pairs of such surfaces forming layers of vanishingly small thickness. In what follows, we consider first the axisymmetrical flow that results from taking the circumferential average of the velocity field due to the vortex and source lines for which analytical results were obtained in the preceding section. We note that this averaging process is equivalent to replacing the periodic delta function, in equations (1) (for vortex lines) and (13) (for source lines), by the first term $N/2\pi$ of its Fourier expansion (16). In turn this is equivalent to distributing the bound vorticity, or the fluid emission by the source lines, uniformly in the circumferential direction, in such a way that we end up with an actuator disc (in the present case with the shape of a spherical annular surface). The theory of the axisymmetric flow due to a plane actuator disc in a cylindrical annular duct is well known [18] and can be considered as a limiting case of the geometry we are dealing with here. In Subsection 3.1 we investigate whether, and under which conditions, the stream surfaces of revolution are exactly conical. In Subsection 3.2 we start from the expressions derived in Section 2 for the potential field due to a row of vortex and source lines in conical flow, and look for the form they take in the limiting case when the angular distance between the walls decreases to zero, i.e. $\theta_2 - \theta_1 \to 0$. The resulting expressions are then modified so that the singularities are isolated and only singularity-free series are left. Finally, in Subsection 3.3, these expressions, combined with the solutions derived in Section 2 for the three-dimensional problem, are used to obtain alternative solutions for the fully three-dimensional flow that are more suitable from the computational point of view.

3.1. Circumferentially averaged velocity field

We will use an overbar to denote a circumferentially averaged quantity, and consider first the case of the vortex lines. Equation (1) gives

$$\bar{\mathbf{V}} = \overline{\nabla\Phi} - \mathbf{e}_\phi H(r-1)\,\frac{\gamma}{R}\,.$$

Here $R = r\sin\theta$ is the distance to the cones' axis, and we have replaced $N\Gamma \equiv 2\pi\gamma$, γ being the disc's circulation per unit circumferential angle (γ/R is the circulation per unit circumferential length). From (3) we find $\overline{\nabla\Phi} = \mathbf{e}_\phi\gamma(2R)^{-1}$, since $\overline{\nabla\Phi_1} = 0$ (as a consequence of $A_{0p}^{(i)} = A_{0p}^{(o)} = 0$, $p = 0, 1, 2, \ldots$) and $\bar{S} = 0$. We then obtain

$$\bar{\mathbf{V}} = \pm\,\mathbf{e}_\phi\,\frac{\gamma}{R}\,. \tag{21}$$

As should be expected, this represents free-vortex flow, with opposite directions of swirl on each side of the disc. If a velocity field of the type $\mathbf{e}_r \times \text{constant}/r^2$ (source or sink at the origin) and/or a field $\mathbf{e}_\phi \times \text{constant}/R$ were superposed upon (21), the resulting flow would still have conical surfaces as stream-surfaces. This situation corresponds, in the cylindrical geometry, to flow in radial equilibrium (zero radial velocity) on both sides of a plane actuator disc. This is known to occur [18] when, as here, the disc introduces a discontinuity in the tangential velocity that is inversely proportional to the radial co-ordinate R. We should not expect in general to have $\bar{V}_\theta = 0$ (conical stream-surfaces) if Γ (or γ) is not spanwisely constant; in such case the distributed trailing vorticity will produce rotational flow downstream of the disc.

We consider next the case of source lines and write $NQ \equiv Q^*$ (total flow rate emittted by the disc) and $Nq \equiv 2\pi q^*$ ($q^* =$ flow rate per unit area of disc). From (14), (5) and (19), we find

$$\bar{\mathbf{V}} = \mp\,\mathbf{e}_r\,\frac{Q^*}{4\pi r^2(y_1 - y_2)} + \overline{\nabla\Phi_1}\,, \tag{22}$$

$$\overline{\nabla\Phi_1} = -\frac{N}{4\pi}\sum_{n=1}^{\infty}\frac{a_{0p}}{\alpha_{0p}}\,r^{\pm\alpha_{0p}-3/2}[\mathbf{e}_r(\pm\alpha_{0p} - \tfrac{1}{2})T_{0p}(y) - \mathbf{e}_\theta\sin\theta\,T'_{0p}(y)]\,. \tag{23}$$

In the special case when $q/\sin\theta = \text{constant}$, we have $a_{0p} = 0$ (see the text following (18)), and so $\overline{\nabla\Phi_1} = 0$ and $\bar{V}_\theta = 0$, i.e., we have conical streamsurfaces. This happens when the source strength q is proportional to the circumferential pitch $(2\pi/N)\sin\theta$, which roughly corresponds to spanwisely constant blockage ratio (blade thickness divided by cascade pitch).

3.2. *Flow in a thin conical layer*

In the quasi-three-dimensional approach outlined at the beginning of this section, the two-dimensional blade-to-blade flow results from dividing the duct space into infinitely thin layers by the family of stream-surfaces of revolution of the circumferentially averaged flow. We saw in Subsection 3.1 that such stream-surfaces are exactly of conical shape when: (i) the line vortices are of constant strength; (ii) the line-source strength is proportional to the distance from the axis; it is obvious that the same will still be true if line-vortices and line-sources satisfying these conditions are distributed over $r_1 < r < r_2$ to model blades of finite chord, and when free-vortex flow or flow due to a point source or sink at the origin are superposed. In what follows, we give expressions for the velocity potential when the angular distance between the conical walls becomes infinitely small, i.e. $\theta_2 = \theta_1 + \varepsilon$, $y_2 = y_1 - \varepsilon'$, $\varepsilon, \varepsilon' \to 0$.

We first note that Legendre's associated equation (6) becomes a differential equation with constant coefficients. For each n, only the zero-order term T_{n0} is kept, and we find

$$T_{n0} = t_{n0} = 1 , \qquad \alpha_{n0} = \left[\frac{1}{4} + \frac{n^2 N^2}{1 - y^2} \right]^{1/2} \equiv \alpha_n \quad (n = 0, 1, 2, \ldots) ,$$

where the subscript 1 has been dropped from y_1.

For a row of vortices, the velocity field is still given by (1) and (3), and the expression for Φ_1 can be found to be

$$\Phi_1 = \pm \frac{\Gamma}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 \pm \frac{1}{2\alpha_n} \right) \sin(nN\phi) r^{\pm \alpha_n - 1/2} . \tag{24}$$

For a row of sources, the potential Φ (which we denote by Φ^{2d}) becomes

$$\Phi^{2d} = \pm \frac{Nq}{4\pi \sin\theta} \left(\frac{1}{r} - 1 \right) - \frac{Nq}{2\pi \sin\theta} \sum_{n=1}^{\infty} \frac{1}{\alpha_n} r^{\pm\alpha_n - 1/2} \cos(nN\phi) . \tag{25}$$

As mentioned before in Section 2, the velocity field has singular behaviour at the vortex or source lines. This results from the superposition of two kinds of singularities. The first one is essentially two-dimensional and inversely proportional to distance, whereas the second one is logarithmic and connected with the curvature of the lines. It is easy to conclude that these singularities result from the contribution of the series in (24) and (25).

We start by dealing with the first type, and note that the singularity is essentially the same as in the two-dimensional plane field about a rectilinear vortex of strength Γ or a rectilinear source of flow rate q per unit length. To separate this two-dimensional contribution, we develop the conical surface θ = constant onto the plane $\zeta = \rho\, e^{i\psi}$, with the vortices or sources lying at $\rho = 1$, $\psi = 2\pi n/N^*$ ($n = 0, \pm 1, \pm 2, \ldots$), where $N^* = N/\sin\theta$. The two-dimensional complex potential due to such a row of point vortices of circulation Γ can be obtained from the well known expression for the rectilinear infinite row (see e.g. [19]) by conformal transformation, and is found to be

$$w(\zeta) = -\frac{i\Gamma}{2\pi} \ln(\zeta^{N^*/2} - \zeta^{-N^*/2}) . \tag{26}$$

To relate this plane field to the conical flow field, we take $\rho = r$, $\psi = -\phi \sin\theta$. Hence the two-dimensional potential (i.e. the real part of w) can be written as

$$\psi = \pm \frac{\Gamma}{2\pi} \left[\arctan \frac{\sin N\phi}{\exp(N^* |\ln r|) - \cos N\phi} + \frac{N\phi}{2} \right] . \tag{27}$$

Using the expansion [20],

$$\arctan \frac{\sin\beta}{e^{\xi} - \cos\beta} = \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\xi} \sin n\beta \quad (\xi > 0) , \tag{28}$$

we then obtain the following expression for the complete potential Φ (which we denote here by Φ^{2d}) due to a row of vortices between two conical walls θ, $\theta + d\theta$:

$$\Phi^{2d}(r, \psi) = \Psi(r, \psi) + \Gamma\Lambda(r, \psi) , \tag{29}$$

where

$$\Lambda(r, \psi) = \mp \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(nN^* \psi)\left[\left(1 \pm \frac{1}{2\alpha_n}\right) r^{\pm\alpha_n - 1/2} - r^{\pm nN^*}\right]. \tag{30}$$

Correspondingly, we find for a row of sources the complex potential

$$w = \frac{q}{2\pi} \ln(\zeta^{N^*/2} - \zeta^{-N^*/2}), \tag{31}$$

whose real part is

$$\Psi = \pm \frac{q}{2} \{\ln[\exp(-2N^*|\ln r|) - 2\exp(-N^*|\ln r|)\cos N\phi + 1] + N^* \ln r\}. \tag{32}$$

Using the expansion [20],

$$\ln(\mathrm{e}^{-2\xi} - 2\,\mathrm{e}^{-\xi} \cos\beta + 1) = -2 \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{e}^{-n\xi} \cos n\beta \quad (\xi > 0), \tag{33}$$

we obtain for a row of sources

$$\Phi^{2d}(r, \psi) = \Psi(r, \psi) \pm \frac{N^* q}{4\pi}\left(\frac{1}{r} + \ln r - 1\right) + q\Lambda(r, \psi), \tag{34}$$

where

$$\Lambda(r, \psi) = -\frac{N^*}{2\pi} \sum_{n=1}^{\infty} \cos(nN^*\psi)\left[\frac{r^{\pm\alpha_n - 1/2}}{\alpha_n} - \frac{r^{\pm nN^*}}{nN^*}\right]. \tag{35}$$

We point out that the second term on the right-hand side of (34) (which is a function of r only) accounts for the fact that the velocity field induced by a point source (at the origin) in plane flow is proportional to (distance)$^{-1}$, differently from the proportionality to (distance)$^{-2}$ for a point source in three dimensions.

It is important to note that, in our two-dimensional space (r, ψ), it is $\nabla^2\Psi = 0$ (except at the sources), but $\nabla^2\Phi^{2d}$ is in general *not* equal to zero. This means that the plane flow represented by Φ^{2d} is not incompressible, a conclusion that should be expected since we are dealing with a layer of fluid whose thickness is not constant (it is proportional to r).

It is not difficult to conclude that Ψ is the only term on the right-hand side of (29) or (34) that contributes to the singularities of type (distance)$^{-1}$ in the velocity field about the vortex and source lines. However, it should not be forgotten that, in spite of the angular width of the conical duct having been reduced to a vanishingly small value, these lines retain their curvature, and so an additional logarithmic singularity is expected to occur at each line, as shown by Küchemann and Weber [17]; it affects the circumferential velocity component V_ϕ in the case of vortices and the radial component V_r for sources. It is obvious that the logarithmic singularities must come from the contribution of the term Λ. Expressions suitable for computation are given in the Appendix for $\partial\Lambda/\partial\psi$ (vortices) and $\partial\Lambda/\partial r$ (sources) in which the logarithmic singularities are separated from the series.

3.3. *Alternative expressions for the three-dimensional potential*

The fact that the velocity field about the vortex or source lines in the fully three-dimensional case ($\theta_2 - \theta_1$ finite) has singularities that are essentially identical to those in a thin conical layer of angular width $d\theta$, suggests that the three-dimensional potential $\Phi(r, \theta, \phi)$ could be expressed as the superposition of the corresponding two-dimensional potential Φ^{2d} (expressions for which were derived in the preceding subsection) and a singularity-free additional field representing the three-dimensionality. To do that, we consider the expressions derived in Section 2 for the three-dimensional potential, as well as equations (24) and (25) for the two-dimensional case. Then, using the second equation (11), we obtain easily, for vortices,

$$\Phi(r, \theta, \phi) = \Phi^{2d} \pm \frac{\Gamma}{2\pi} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{t_{np}}{n} \sin(nN\phi) T_{np}(y) \left[\left(1 \pm \frac{1}{2\alpha_{np}}\right) r^{\pm\alpha_{np}-1/2} - \left(1 \pm \frac{1}{2\alpha_n}\right) r^{\pm\alpha_n - 1/2} \right]. \tag{36}$$

For sources, we use (17) and find

$$\Phi(r, \theta, \phi) = \Phi^{2d} - \frac{N}{4\pi} \sum_{p=1}^{\infty} a_{0p} T_{0p}(y) \left[\frac{r^{\pm\alpha_{0p}-1/2}}{\alpha_{0p}} \pm \left(\frac{1}{r} - 1\right) \right] - \frac{N}{2\pi} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} a_{np} \cos(nN\phi) T_{np}(y) \left[\frac{r^{\pm\alpha_{np}-1/2}}{\alpha_{np}} - \frac{r^{\pm\alpha_n-1/2}}{\alpha_n} \right]. \tag{37}$$

Considering what was said above, it is expected that the series on the right-hand side of (36) and (37) will present no convergence problems, even in the vicinity of the vortex and source lines.

4. Applications

The results derived above for the velocity field $\mathbf{V}(r, \theta, \phi)$ can be generalized to obtain the velocity $\mathbf{V}^*(r, \theta, \phi)$ due to a row of vortex lines of circulation Γ or to a row of source lines of flow rate $q(\theta)$ per unit length of line, located at $r = r_0$, $\phi = \phi_0 + 2\pi n/N$ ($n = 0, 1, 2, \ldots, N-1$). We easily find

$$V_r^*(r, \theta, \phi) = r_0^{-1} V_r(r/r_0, \theta, \phi - \phi_0), \tag{38}$$

with identical relations applying to the components V_θ and V_ϕ.

In most applications, the flow field far upstream (i.e., as $r \to 0$ for outward flow or $r \to \infty$ for inward flow) is given, and a singularity distribution satisfying certain conditions (given blade geometry, flow deflection, etc.) is to be found. In such cases it is convenient to add $\pm N\Gamma(4\pi r \sin\theta)^{-1}$ to the circumferential velocity component V_ϕ as given by the expressions derived in Sections 2 and 3 for the flow field due to a row of vortex lines. (Here, the plus sign is to be taken for inward flow and the minus sign for outward flow.) In this way, it can easily be found that the resulting velocity $\mathbf{V}(r, \theta, \phi)$ due to a row of vortex lines vanishes far upstream. We note that no similar modification is needed for the case of source lines, since the condition of zero total flow rate of the source distribution has to be satisfied anyway if blades with a closed surface are to be modelled. Then the complete velocity field can be considered as the superposition of: (i) a basic field, due to a source at the origin and possibly a vortex along the duct's axis, coinciding with the incoming field far upstream; and (ii) a perturbation field due to a distribution of vortex and source lines on or within the blades' surfaces.

In order to illustrate the application of the expressions derived above to a case of practical interest, we consider the flow about the inlet guide vane system of a water turbine of bulb type, consisting of N blades spanning two coaxial conical walls with a common vertex. For simplicity, the distance between the walls is assumed very small, so that the expressions derived in Section 3.2 may be applied. In the panel method used for the numerical calculation, the blade profile is approximated by a contour consisting of n rectilinear segments. Vortices and sources are uniformly distributed over each line segment. (For a general description of the method see [21].) In order to make the problem determinate, the vortex distribution density is taken equal for all segments. The resulting velocity field is given by the superposition of the undisturbed flow field and the fields induced by the singularity distributions on the blade contour. The vortex and source distribution densities at the n line-segment elements ($n+1$ unknowns) are then determined by introducing: (i) the condition of velocity tangent to the contour at n control points chosen as the mid-points of the elements; and (ii) the Kutta condition at the sharp trailing edge, in the form of equality of velocities at the control points on the two segments adjoining the trailing edge. The use of the expressions derived in Section 3.2 ensures that the boundary conditions at the conical walls, as well as the condition of circumferential periodicity, are automatically satisfied.

The computation process involves the calculation of the velocity at each control point j ($j = 1$ to n) induced by the vortex or the source unit-density distribution on the contour element k ($k = 1$ to n); such values form a pair of matrices of influence coefficients. These coefficients are obtained by integrating over the contour element in question the expressions for the velocity field induced by a row of vortex or source lines of unit strength; such expressions can be obtained from the results of Section 3.2 generalized by transformation (38). If $j = k$ (velocity induced by a contour segment at its own control point), the integrand is singular at the control point, and the principal value of the integral is to be taken. As referred to above, the velocity field presents two kinds of singularities. The stronger one, of type $(\text{distance})^{-1}$, is essentially two-dimensional and results from taking the gradient of the right-hand side of (27) or (32); it can easily be separated and integrated analytically (for $j = k$, as well as for $j \neq k$, see [21]). The weaker, logarithmic singularity is essentially three-dimensional and was separated in the Appendix. The separated logarithmic function can be integrated analytically for $j = k$, and integrated numerically or simply taken constant for $j \neq k$. The remaining terms in the expressions of the velocity field are regular and may be taken constant in the integration with good approximation.

Numerical results were computed for the conical cascade represented unfolded in Fig. 2. The cone angle is $\theta = 25°$, and the cascade comprises six blades of NACA 63A604 profile. The incoming flow is directed towards the cone vertex and is swirl-free. Figure 3 shows numerical results (solid line), obtained with 400 line-segment elements along the blade contour, for the blade surface pressure coefficient (defined as $1 - V^2 V_0^{-2}$) as a function of distance along the blade chord. Here V is the calculated velocity on the blade surface, and V_0 (proportional to r^{-2}) is the velocity of the undisturbed flow at the same location. Also shown in Fig. 3 for comparison are the results (dotted line) for the corresponding two-dimensional plane flow about the circular cascade of blades depicted in Fig. 2 (in this case V_0 is proportional to r^{-1}); these results were obtained by keeping only the two-dimensional term Ψ in the right-hand side of (29) and (34). The distance between the solid and dotted lines in Fig. 3 can be regarded as representing the three-dimensional effects due to radially-varying

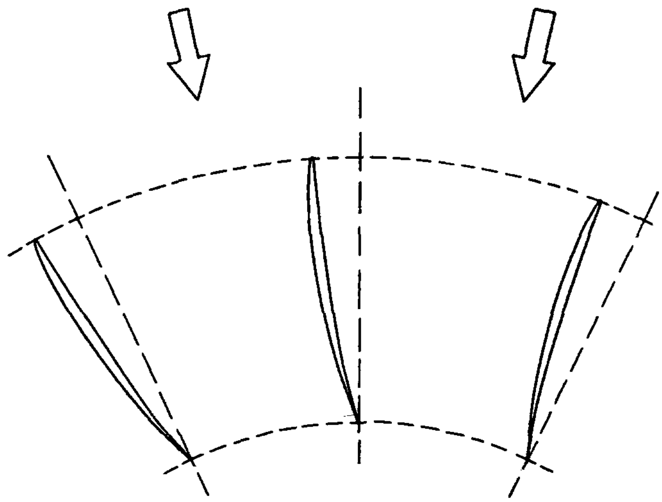

Fig. 2. Geometry of unfolded conical cascade comprising six NACA 63A604 profile blades. Cone angle is $\theta = 25°$.

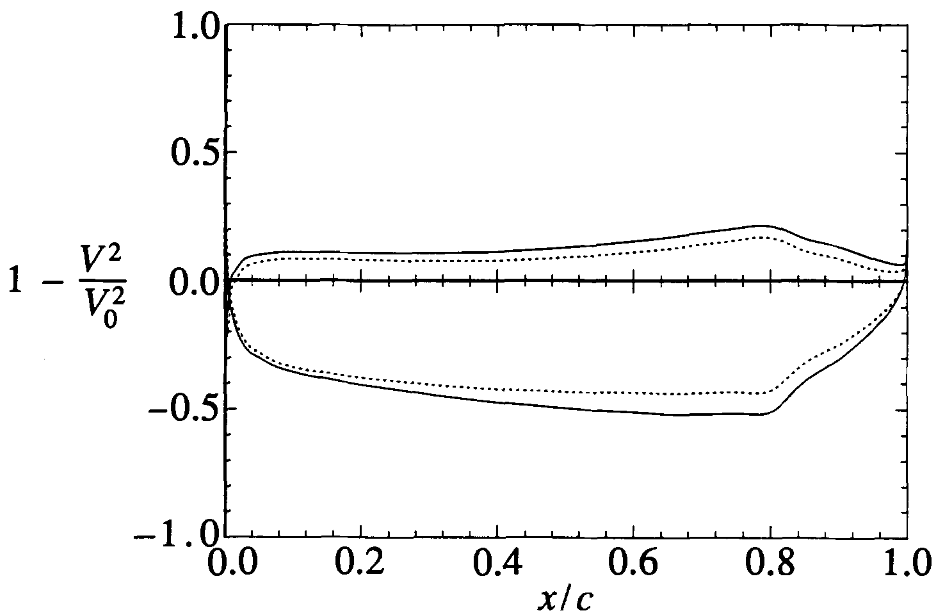

Fig. 3. Blade surface pressure coefficient versus distance along the chord for the cascade of Fig. 2. V and V_0 are the surface velocity and the velocity of the undisturbed flow respectively. The dotted line represents results from simple two-dimensional theory.

distance between the conical walls and to singularity-line curvature; the figure shows that such effects can be substantial.

5. Conclusions

Analytical expressions in the form of double series have been derived for the velocity field due to vortex and source lines in a conical annular duct. In both cases it was possible to isolate the mathematical singularities of type $(\text{distance})^{-1}$ in a two-dimensional term expressed in closed form. The weaker, logarithmic singularities associated with vortex and source line curvature were also isolated. In this way convergence problems were overcome, rendering the expressions suitable for integration over the blade chord, or over the blade surface elements if a panel method is used. In the limiting case of axisymmetric flow (actuator disc), it has been established that the stream-surfaces of revolution are of exact conical shape if the vortex lines that make up the disc are of spanwisely constant strength, or, for source flow, if the fluid is emitted at a uniform rate per unit area of the disc (spherical) surface. This can be regarded as an extension of the corresponding results for a classical plane actuator disc in a cylindrical duct. We point out that the expressions derived for the other limiting, two-dimensional case (very small distance between the conical walls) are suitable for use in blade-to-blade-flow numerical methods, as part of a quasi-three-dimensional calculating scheme or when the distance between the walls is considered small enough to make such approximation acceptable. This was illustrated by a numerical example in which the analytical expressions derived here were combined with a panel method to compute the flow about a conical row of blades.

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Appendix. Singularities of $\partial\Lambda/\partial r$, $\partial\Lambda/\partial\psi$

We consider first the case of vortices, and rewrite equation (30) in the form

$$\Lambda = \mp\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(nN^*\psi)[b_n(r)+c_n(r)]\,, \tag{A1}$$

where

$$b_n(r) = r^{\pm nN^*}\left[\pm\frac{1}{2nN^*} + \left(\mp\frac{1}{8nN^*} - \frac{1}{2}\right)\ln r\right]. \tag{A2}$$

Applying well-known techniques of expansion in power series, it can be shown that

$$c_n(r) = r^{\pm nN^*}[\mathrm{O}(nN^*)^{-3} + \mathrm{O}(\ln^2 r)]\,, \quad \text{as } n\to\infty \text{ and } r\to 1\,. \tag{A3}$$

Then, if use is made of (28) and (33), we obtain

$$\frac{\partial\Lambda}{\partial\psi} = \left(1 - \frac{1}{4}\ln r\right)A + B \mp \frac{N^*}{2\pi}\sum_{n=1}^{\infty}\cos(nN^*\psi)c_n(r)\,, \tag{A4}$$

where

$$A = \frac{1}{8\pi}\ln(r^{\pm 2N^*} - 2r^{\pm N^*}\cos N^*\psi + 1)\,, \tag{A5}$$

$$B = \pm \frac{N^*}{4\pi} \ln r \, \frac{r^{\mp N^*} \cos N^* \psi - 1}{r^{\mp 2N^*} - 2r^{\mp N^*} \cos N^* \psi + 1} \,. \tag{A6}$$

It can be found that A becomes infinite and B of the form $0 \times \infty$, as $r \to 1$ and $\psi \to 0$. Then we write $r = 1 + \varepsilon \cos \alpha$, $\psi = \varepsilon \sin \alpha$, and obtain

$$A = \frac{1}{4\pi} \ln(N^* |\varepsilon|) + O(N^* \varepsilon) \,, \qquad B = -\frac{1}{4\pi} \cos^2 \alpha + O(N^* \varepsilon) \,, \quad \text{as } \varepsilon \to 0 \,. \tag{A7}$$

In the same way, for sources, we start from (35), and write

$$\frac{\partial \Lambda}{\partial r} = -\frac{N^*}{2\pi r} \sum_{n=1}^{\infty} \cos(nN^* \psi)[d_n(r) + e_n(r)] \,, \tag{A8}$$

where

$$d_n(r) = r^{\pm nN^*} \left[-\frac{1}{2nN^*} + \left(\frac{3}{8nN^*} \mp \frac{1}{2} \right) \ln r \right] ; \tag{A9}$$

it can be shown that, as $n \to \infty$ and $r \to 1$, $e_n(r)$ is of the same order of magnitude as $c_n(r)$ (see (A3)). Then we find

$$\frac{\partial \Lambda}{\partial r} = -\frac{A}{r} \left(1 - \frac{3}{4} \ln r \right) + \frac{B}{r} - \frac{N^*}{2\pi r} \sum_{n=1}^{\infty} \cos(nN^* \psi) e_n(r) \,, \tag{A10}$$

where A and B are still given by (A5) and (A6).

The series in (A4) and (A10) present no computational difficulty even at, and in the vicinity of, the vortices and sources.

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